Sunday, March 01, 2015 7:57 PM

Flatness

We would like a coordinate invariant way of saying a space is flat.

i) We could look at the metric, but even in flat space the metric can look funny if we choose the wrong coordinates.

Recall $g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$ so even if $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we can get crazy $g_{\mu'\nu'}$.

ii) We do know that in flat space w/ cartesian coordinates $\Gamma^\mu{}_{\nu\lambda} = 0$, so we could say this is a condition for flatness.

Note: This might work if Γ was a tensor since "$\underbrace{\Gamma^{\mu'}{}_{\nu'\lambda'}}_{=0} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\underbrace{\Gamma^\mu{}_{\nu\lambda}}_{=0}$" $\Leftarrow$

but it ain't:

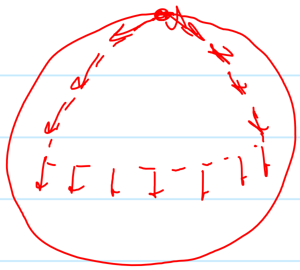
$$\Gamma^{\mu'}{}_{\nu'\lambda'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\underbrace{\Gamma^\mu{}_{\nu\lambda}}_{=0} - \frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\lambda}$$

not zero!

iii) Also remember that at any point we can always choose LIC's so that $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Gamma = 0$, even if the space is curved!

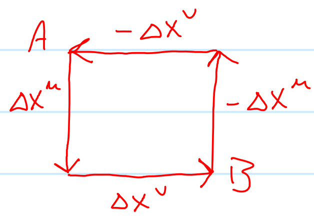
What are we to do?!

Recall:



If we $\parallel$-transport any vector around a closed path in curved space then it generally comes back changed.

Okay, so let's choose an infinitesimal closed path:

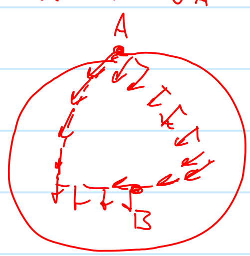


We could start w/ a vector at A and then transport it:

$(-\hat{\Delta X}^\nu)(-\hat{\Delta X}^\mu)\hat{\Delta X}^\nu \hat{\Delta X}^\mu V_A^\lambda = \tilde{V}_A^\lambda$ and then compute $\tilde{V}^\lambda - V^\lambda = \delta V^\lambda$ (=0 for flat)

Or we could take the vector from A to B along two paths:

$\hat{\Delta X}^\nu \hat{\Delta X}^\mu V_A^\lambda = \tilde{V}_B^\lambda$, $\hat{\Delta X}^\mu \hat{\Delta X}^\nu V_A^\lambda = \tilde{\tilde{V}}_B^\lambda$ then compute $\tilde{V}_B^\lambda - \tilde{\tilde{V}}_B^\lambda = \delta V^\lambda$ (=0 for flat)



But what operators $\hat{\Delta X}$ can we use to $\parallel$-transport a vector around?

How about ∇_μ?

So let's compute the commutator of two covariant derivatives as they act on vectors:

$$[\nabla_\mu, \nabla_\nu] V^\lambda = \nabla_\mu \underbrace{\nabla_\nu V^\lambda}_{T_\nu{}^\lambda} - \nabla_\nu \nabla_\mu V^\lambda$$

$T_\nu{}^\lambda$ so when we act w/ ∇_μ we need 2 Christoffels

$$= \partial_\mu (\underbrace{\nabla_\nu V^\lambda}_{(\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\rho} V^\rho)}) - \Gamma^\alpha_{\mu\nu} \nabla_\alpha V^\lambda + \Gamma^\lambda_{\mu\alpha} \nabla_\nu V^\alpha - (\mu \leftrightarrow \nu)$$

$$= \underline{\partial_\mu \partial_\nu V^\lambda} + (\partial_\mu \Gamma^\lambda_{\nu\rho}) V^\rho + \underline{\Gamma^\lambda_{\nu\rho} \partial_\mu V^\rho} - \Gamma^\alpha_{\mu\nu} \partial_\alpha V^\lambda$$

$$- \Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\alpha\rho} V^\rho + \underline{\Gamma^\lambda_{\mu\alpha} \partial_\nu V^\alpha} + \Gamma^\lambda_{\mu\alpha} \Gamma^\alpha_{\nu\rho} V^\rho - (\mu \leftrightarrow \nu)$$

The underlined terms will cancel when combined w/ $(\mu \leftrightarrow \nu)$ subtraction leaving:

$$[\nabla_\mu, \nabla_\nu] V^\lambda = \underbrace{\underbrace{(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho}}_{\partial^2 g} + \underbrace{\Gamma^\lambda_{\mu\alpha} \Gamma^\alpha_{\nu\rho} - \Gamma^\lambda_{\nu\alpha} \Gamma^\alpha_{\mu\rho}}_{(\partial g)^2}}_{\text{good measures of curvature (do not vanish even in LICs!)}}) V^\rho - 2 \Gamma^\alpha_{[\mu\nu]} \nabla_\alpha V^\lambda$$

Recall $\Gamma \sim \partial g \Rightarrow$

$$[\nabla_\mu, \nabla_\nu] V^\lambda = \underbrace{(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\nu\rho} - \Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\mu\rho})}_{\equiv R^\lambda{}_{\rho\mu\nu}} V^\rho$$

But it gets better because this thing is a tensor, because we are just taking "good" derivatives of tensors.

In 4D this could have $4^4 = 256$ independent components.

But we can show (by first forming $R_{\alpha\rho\mu\nu} = g_{\alpha\lambda} R^\lambda{}_{\rho\mu\nu}$):

$R_{\alpha\rho\mu\nu} = -R_{\alpha\rho\nu\mu}$ (antisymmetry in last 2 which follows from $[\nabla_\mu, \nabla_\nu] = -[\nabla_\nu, \nabla_\mu]$)

$R_{\alpha\rho\mu\nu} = -R_{\rho\alpha\mu\nu}$ (antisymmetry in first 2)

$R_{\alpha\rho\mu\nu} = R_{\mu\nu\alpha\rho}$ (symmetry under $2 \leftrightarrow 2$)

$R_{\alpha\rho\mu\nu} + R_{\alpha\nu\rho\mu} + R_{\alpha\mu\nu\rho} = 0$ (cyclic reorder of last 3)

All of these symmetries reduce $256 \to 20$ independent components!

Flatness

If $R^\lambda{}_{\rho\mu\nu} = 0$, then the space is flat. Note, since $R^\lambda{}_{\rho\mu\nu}$ is a tensor, this is coordinate invariant, i.e. $\underbrace{R^{\lambda'}{}_{\rho'\mu'\nu'}}_{=0} = \frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\rho}{\partial x^{\rho'}}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\underbrace{R^\lambda{}_{\rho\mu\nu}}_{=0}$

There are other degrees of flatness which can be specified w/ objects built from $R^\lambda{}_{\rho\mu\nu}$.

First a quick useful fact: $T^{(\mu\nu)}W_{[\mu\nu]} = 0$

2D example: $T^{(\mu\nu)}W_{[\mu\nu]} = T^{00}\underbrace{W_{00}}_{=0} + T^{01}W_{01} + \underbrace{T^{10}W_{10}}_{-T^{01}W_{01}} + T^{11}\underbrace{W_{11}}_{=0}$

Consider all 2-index contractions of $R^\lambda{}_{\rho\mu\nu}$:

i) $R^\lambda{}_{\lambda\mu\nu} = g^{\lambda\alpha}R_{\alpha\lambda\mu\nu} = g^{(\lambda\alpha)}R_{[\alpha\lambda]\mu\nu} = 0$

ii) $R^\lambda{}_{\rho\lambda\nu} = g^{\lambda\alpha}R_{\alpha\rho\lambda\nu} = R_{\rho\nu}$ <u>Ricci Tensor</u>

iii) $R^\lambda{}_{\rho\mu\lambda} = -R^\lambda{}_{\rho\lambda\mu} = -R_{\rho\mu}$ (same as Ricci Tensor)

Note: $\underbrace{R_{\mu\nu} = R_{\nu\mu}}_{\text{Ricci is symmetric}}$ since $R_{([\alpha\lambda][\mu\nu])}$.

We can then form: $R = R^\nu{}_\nu = g^{\nu\mu}R_{\mu\nu}$ <u>Ricci Scalar</u>

And then: $R^\lambda{}_{\rho\mu\nu}$ − all contractions $= C_{\rho\sigma\mu\nu}$ <u>Weyl Tensor</u>

Then $\{C_{\rho\sigma\mu\nu}, R_{\mu\nu}, R\}$ contains everything in $R^\lambda{}_{\rho\mu\nu}$.

Tuesday, March 03, 2015 1:39 PM

How flat is flat?

Examples

$R^{\lambda}{}_{\rho\mu\nu} = 0$ Flat-flat $\mathbb{M}^n$, $\mathbb{R}^n$, $\mathbb{T}^n$

$R_{\mu\nu} = 0$ Ricci-flat $AdS^5 \times S^5$

$C_{\rho\sigma\mu\nu} = 0$ $\underbrace{\text{Conformally-flat}}$ All 2D pseudo-Riemannian manifolds (since $C_{\rho\sigma\mu\nu} = 0$)

Can be mapped to flat space w/ conformal transformation (locally angle preserving trans.)

$R = 0$ Doesn't mean much (however for certain maximally symmetric spaces, R completely determines the curvature.)